*International Journal of Theoretical Physics, Vol. 30, No. 7, 1991*

# Finitary Substitute for Continuous Topology

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*Received January 4, 1991*

Finite topological spaces are combinatorial structures that can serve as replacements for, or approximations to, bounded regions within continuous spaces such as manifolds. In this spirit, the present paper studies the approximation of general topological spaces by finite ones, or really by "finitary" ones in case the original space is unbounded. It describes how to associate a finitary space $F$ with any locally finite covering of a $T_1$-space $S$; and it shows how $F$ converges to $S$ as the sets of the covering become finer and more numerous. It also explains the equivalent description of finite topological spaces in order-theoretic language, and presents in this connection some examples of posets $F$ derived from simple spaces $S$. The finitary spaces considered here should not be confused with the so-called causal sets, but there may be a relation between the two notions in certain situations.

## 1. INTRODUCTION

That matter on the smallest scales sheds its continuous nature is indicated by several features of present-day physics. In particular, the short-distance "cutoffs" required (apparently) by both quantum field theory (to "regularize" the functional integral) and "quantum gravity" (to render black hole entropy finite) seem ultimately foreign to the notion of differentiable manifold embodied in classical general relativity. Their stubborn presence suggests, rather, that there is a discrete substratum underlying spacetime and accounting naturally for the appearance of a minimum length in the effective theories we now possess. Such an underlying discreteness, moreover, has often been looked to in the hope of finding explanations for such general features of nature as the existence of the spacetime metric, the presence of gauge and other fields interacting with this metric, the $(3+1)$-dimensionality of spacetime, the directionality of time, and the near vanishing of the cosmological constant.

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0020-7748/91/0700-0923$06.50/0 

Even if spacetime should prove to be a true continuum after all, latticelike approximations to it would no doubt continue to be useful as technical tools. Such approximations have already been employed "calculationally" to study theories such as QCD (Duke and Owens, 1985), but they also offer hope of shedding light on presently obscure conceptual issues of quantum gravity. In particular, the results to be presented below furnish a framework in which one might express the difficult to formulate idea that the topology of spacetime (or space) may be highly ramified and ceaselessly fluctuating on small scales ("foamlike"), and yet "smooth out" on large scales to produce a topologically featureless continuum such as $\mathbf{R}^4$.

Leaving such nonfundamental applications aside for now, let us assume that a discrete substratum really exists and ask how we might expect it to be organized. One possibility, of course, is that it has an entirely unexpected structure unrelated to anything we know in the continuum. In that case, though, it would be hard to foresee how the substratum serves to underpin the continuum at all. A more fruitful assumption at this stage might be that some basic aspect of the substratum's organization is already familiar to us from our study of the continuum—that the continuum resembles the substratum in a definite manner which can help us to mediate between descriptions at the deeper level and descriptions at the more coarse-grained level on which Riemannian (or rather Lorentzian) geometry emerges.

In trying to carry this assumption farther, the first decision we face is which structure or structures of Lorentzian geometry to hang onto as we descend to the level of the substratum. Should it be the topology, the metric structure, the $SL(2, \mathbf{C})$ spin-structure, the causal structure, or something else? All of these possibilities have been proposed, but let me concentrate here on topology. If we assume a topological resemblance between continuum and discrete substratum that is as close as possible, then the latter will literally be a topological space, but one in which any bounded region (in an appropriate sense) comprises only a finite number of elements or "points". In other words, the substratum will be what I will call a "finitary topological space."

A recent invocation of such a model in the context of "quantum topology" may be found in work by Isham (1989*a*,*b*), who shows how finitary topologies (and possibly more general ones) may be quantized along algebraic lines. Basing himself on the lattice of all topologies $\tau$ on a given set $S$, and treating this lattice as a kind of configuration space within which $\tau$ can vary, he defines an abstract $*$-algebra $A$ generated by plausible analogues of position and momentum variables for $\tau$. He then observes that the permutations of $S$ function as gauge transformations of the theory, and shows how to pass to a subalgebra of $A$ which is gauge invariant in this sense. Finally, he studies a specific Hamiltonian which is capable of inducing transitions from one topology $\tau$ to another! Physically, such a model might have more than one application, but in relation to quantum gravity it would seem to fit most directly into a "canonical" approach in which the topological substratum underpins, not spacetime itself, but only "space at a fixed time" (with whatever meaning this last phrase would ultimately acquire).

Another possibility is that the substratum corresponds to spacetime as a whole, with its dynamics being introduced via a sum-over-histories or some other "covariant" procedure. A theory of this general type was proposed by Finkelstein and Rodriguez (1986; Finkelstein, 1987), who, however, chose the simplicial complex rather than the finitary topological space per se as their fundamental mathematical structure. Like Isham, they favor an algebraic approach to "quantization"; but rather than construe the basic topological variables as elements of an ordinary *-algebra, they seek them within an alternatively structured algebra intended to provide a language for an entire quantum set theory from which the special case of the quantum simplicial complex would follow automatically. In this way they attempt to build quantum mechanics in from the outset, and also to endow finitary topological notions with an algebraic content which would allow an $SL(2, \mathbf{C})$ corresponding to physical spin to emerge directly out of the substratum. This attempt to fuse topology with spin produces a framework within which the relation between the macroscopic topology and that of the underlying simplicial complex appears to be much less direct than is the relation between large- and small-scale topology in a model of the Isham type. It is thus unlikely that the considerations of Section 4 below would apply to the theory of Finkelstein and Rodriguez (1986; Finkelstein, 1987) as readily as they might to that of Isham (1989*a*,*b*).

Another way in which finitary topology might become relevant to the discrete substratum would be if the latter possessed a structure which, while not in itself topological, was nevertheless close enough in mathematical character that one could derive a finitary topological space from it by some auxiliary construction, for example, via some sort of coarse-graining. In this connection it is noteworthy that every finitary topological space has an equivalent description as a partially ordered set (poset), as we will see below. For this reason, constructions leading to finitary topological spaces might be useful in theories which attribute a partial ordering to the substratum, even if that ordering is not topological in character.

Thus, frameworks in which the substratum carries a microscopic *causal* order (causal in the sense that it corresponds to the macroscopic relation of before and after) are ones in which finite topological spaces might turn out to be useful, even if they do not come into consideration at the most fundamental level of theory. Among such ideas is the "causal set" hypothesis of Bombelli *et al.* (1987) and Sorkin (1990), which embodies the approach to quantum gravity that I personally favor. A related proposal is put forth in Finkelstein (1988, 1989*a*,*b*), wherein, however, the fundamental order has a strong topological flavor mixed in with its causal one. (Or perhaps a better interpretation might be that the order is neither topological nor causal, but rather one of abstract succession, while the physical topology and metric arise indirectly via an algebraic construction of spacetime in terms of tensors whose index structure mirrors the abstract order.) In contrast, a causal set in the sense of Bombelli *et al.* (1987) and Sorkin (1990) carries no topological structure at all, except insofar as one emerges at the same level of being on which the continuum itself begins to exist.

In this third sort of situation (to which I will return briefly in the concluding section) two distinctly different posets would be involved. The first would be nontopological and fundamental, expressing the most basic organization of the substratum; the second would be topological but derivative, arising from the first by some formal procedure intended to bring out the topological information implied by the latter's more fundamental order. This kind of application of finitary topological spaces would be in a sense intermediate between the two kinds of application discussed above: ones in which the topological poset is itself fundamental, and ones in which it is introduced entirely by hand in order to facilitate the study of what is ultimately a continuum theory from start to finish.

In the present paper I will not try to gear the mathematical development to any one or another of the potential applications broached above. Rather, I will study a question which would likely be important for any one of them: the question of the manner in which a finitary topological space (or sequence of them) can approximate (or in the limit reproduce exactly) a more general topological space such as a manifold. The possibility of such an approximation would seem to be required whether the finitary topological structure is intended as fundamental in itself, as a help in mediating between the substratum and the continuum, or merely as a technical aid in analyzing a true continuum theory.

The material to be presented below [much of which was announced in Sorkin (1983)] begins with a brief introduction to finite topological spaces, showing their relation to posets and to coverings of continuum spaces; it ends with the proof of a theorem on approximating an arbitrary $T_1$-space by finitary spaces. In more detail: Section 2 explains how to associate a finite topological space with a covering of a given space by a finite number of open sets; Section 3 introduces the equivalent description of a finitary topological space as an ordered set (poset) and presents some examples illustrating how such topological notions as connectedness, continuity, and homology become reformulated in order-theoretic terms; and Section 4 proves some theorems showing in what sense the finitary topological space derived from a covering converges to the original as the covering sets become smaller and more numerous.

## 2. THE FINITE TOPOLOGICAL SPACE ASSOCIATED WITH A FINITE COVERING

Let $S$ be any topological space, for example, spacetime or a spacelike hypersurface thereof. From an "operational" perspective, an individual point of $S$ is a very ideal limit of what we can directly measure. A much better correlate of a single "position-determination" would probably be an *open subset* of $S$. Moreover, even for continuum physics, the individual points (or "events") of $S$ exist only as *carriers* for the topology, and thereby also for higher-level[2] constructs such as the differentiable structure and the metric and "matter" fields: not the points per se, but only this kind of relation involving them has physical meaning.

Recall now that mathematically the topology $\mathfrak{J}$ of $S$ is precisely a family of subsets of $S$ (the *open* subsets). Thus, if for some reason we have access to only a finite number of open sets (for example, those "given" by our previous measurements), then we have in effect access not to the full topology $\mathfrak{J}$, but only to a subtopology $\mathcal{U} \subset \mathfrak{J}$. In such a situation the above remarks suggest we can codify our topological knowledge of $S$ in the structure of the space $F(\mathcal{U})$ obtained from $S$ by *identifying* with each other any two of its points which are not distinguished by the sets of $\mathcal{U}$.

More formally, let $\mathcal{U} \subset \mathfrak{J}$ be a collection of open sets whose union is $S$ (i.e., an open cover of $S$). Assume that $\mathcal{U}$ is finite (whence $S$ is best compared with a *bounded* region of space or spacetime) and that it is closed under the operations of union and intersection, thus forming a subtopology on $S$. Regard $x, y \in S$ as equivalent iff $\forall U \in \mathcal{U}, x \in U \Leftrightarrow y \in U$. Let $F(\mathcal{U})$ be the quotient of $S$ with respect to this equivalence and let $f(\mathcal{U}): S \to F(\mathcal{U})$ be the map taking $x \in S$ into the equivalence class to which it belongs. By definition, a subset $A \subset F(\mathcal{U})$ will be open iff the union of the equivalence classes comprising $A$ is one of the open sets $U$, i.e., iff[3] $f(\mathcal{U})^{-1}(A) \in \mathcal{U}$. Equivalently, the open sets in $F$ are those of the form $f(\mathcal{U})[U]$ for $U \in \mathcal{U}$.

Consider, for example, the simple situation of Figure 1. Here there might have been two imprecise and overlapping "point determinations" (the two disks) giving rise to a covering of $S = A \cup B \cup C$ by the open sets $A \cup C$, $B \cup C$, and $C$ (as well as the empty set and $S$ itself, which gives $\mathcal{U}$

[2]In case $S$ is spacetime, this assignment of "levels" is not unambiguous. One can equally regard the causal structure as basic, with the topology and differentiable structure being derived from it.

[3]Notice that this differs in general from another possible definition which would require only membership in $\mathfrak{J}$.

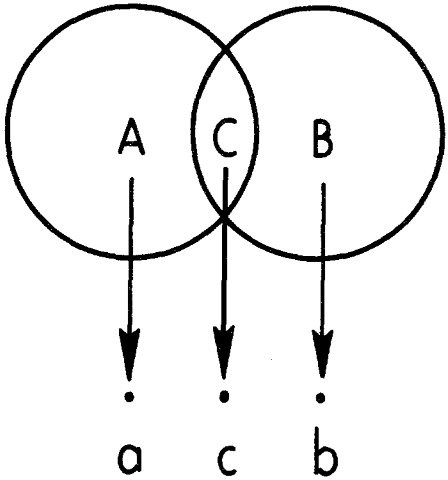



**Fig. 1.** A covering and its $T_0$-quotient.

a total of five members). The associated finite space $F = F(\mathcal{U})$ has three points: $F = \{a, b, c\}$; and its nontrivial open subsets are $\{a, c\}, \{b, c\}, \{c\}$. It is clear that $F(\mathcal{U})$ carries precisely as much topological information about $S$ as $\mathcal{U}$ provides.[4]

There is a more abstract description of the above procedure which will be useful to us in considering how $F(\mathcal{U})$ converges to $S$ as $\mathcal{U}$ grows finer. A topological space $S$ is a $T_0$-space if for any pair of distinct points of $S$ there is an open set containing one point and not the other. (With this definition the above remark that points are no more than *carriers* of topology can be phrased by asserting that every physical space is a $T_0$-space.) Now, if some space $X$ is *not* $T_0$, it can be "made so" by identifying unseparated points just as we did above, where we in effect constructed $F(\mathcal{U})$ as the $T_0$-quotient of $S$ with respect to the topology $\mathcal{U}$.

This quotient can be defined by the universal mapping property illustrated in Figure 2. One calls $f: X \to Y$ *universal* among maps into $T_0$-spaces iff $Y$ itself is $T_0$, and for any other map $g$ of $X$ into a $T_0$-space $Z$, there is a *unique* $k$ such that $g = kf$ (here "map" means "continuous function"). The statement that $f: X \to Y$ is universal in this sense *defines* $Y$ (up to homeomorphism) as the $T_0$-quotient of $X$. For completeness, let us check that $f = f(\mathcal{U})$ as defined above actually solves this "universal mapping problem", i.e., that it "makes $X = (S, \mathcal{U})$ into a $T_0$-space".

*Lemma 2.1.* With $S, \mathcal{U}$ as above, the projection $f(\mathcal{U}): S \to F(\mathcal{U})$ is universal among $\mathcal{U}$-continuous maps of $S$ into $T_0$-spaces.

[4]In this sense it refines the notion of the "nerve" of a covering (Aleksandrov, 1956).

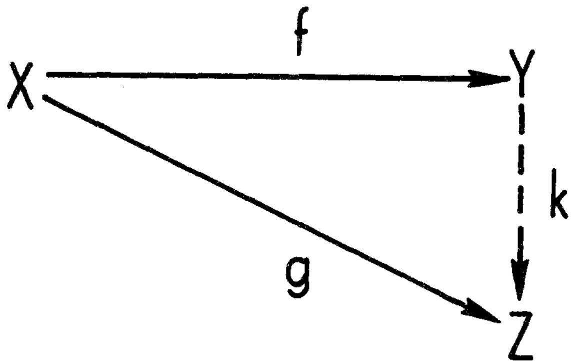



**Fig. 2.** Diagram defining $Y$ as the $T_0$-quotient of $X$. In the application to finite coverings, $X=(S, \mathscr{U})$, $Y=F(\mathscr{U})$, and $f=f(\mathscr{U})$.

*Proof.* Referring to Figure 2, let $X$ be $(S, \mathscr{U})$ (i.e., $S$ provided with the topology $\mathscr{U}$) and let $f=f(\mathscr{U})$, $Y=F(\mathscr{U})$. Also let $[x]:=f(x)$ be the equivalence class of $x$ as defined above, i.e., the set of points of $S$ not separated from $x$ by $\mathscr{U}$. Since $f$ is surjective (obviously), the required $k$ will be unique if it exists at all and must be given by $k([x])=g(x)$. This specification of $k$ will be consistent as long as $[x]=[x'] \Rightarrow g(x)=g(x')$. But if $g(x) \neq g(x')$, then since $Z$ is $T_0$, there is an open set $W$ separating $g(x)$ from $g(x')$. Hence $g^{-1}[W]$ on one hand belongs to $\mathscr{U}$ (by the definition of $g$) and on the other hand clearly separates $x$ from $x'$, which therefore are inequivalent: $[x] \neq [x']$. Thus, $k$ is well defined and satisfies $g=kf$ by construction. Further, $k$ is continuous. For given any open $W \subset Z$, we have $f^{-1}[k^{-1}[W]]=(kf)^{-1}[W]=g^{-1}[W]$, which is open by the continuity of $g$; hence $k^{-1}[W]$ is open by the definition of the topology of $Y=F(\mathscr{U})$. Finally, let us verify that $F(\mathscr{U}) \equiv Y$ is itself $T_0$. If $[x] \neq [y]$, then by definition there is $U \in \mathscr{U}$ for which (say) $x \in U$, $y \notin U$. This implies in turn that $\forall z \in U$, $[z] \neq [y]$, whence $f(y)=[y] \notin f[U]$. Hence $f[U]$ (which is open by definition) separates $[x]$ from $[y]$. ■

For future use, let us record the just-used fact that, as an immediate consequence of the definition of $F(\mathscr{U})$ in terms of equivalence classes,

$$f(\mathscr{U})^{-1} f(\mathscr{U})[U]=U \quad \text{for any } U \in \mathscr{U} \tag{2.1}$$

In replacing $S$ by $F(\mathscr{U})$, we have in effect approximated one topological space by another one of a very special type: a topological space containing only a finite number of elements. In Section 4 we will see in what sense the approximation improves as more and more open sets are added to $\mathscr{U}$.

## 3. ORDER AND TOPOLOGY (POSETS)

Let us pause to consider the notion of finite topological space in its own right, before taking up the approximation question again in the next section. By definition such a space is a finite set $F$ together with a collection $\mathfrak{I}$ of subsets of $F$ (the *open* sets) closed under union and finite intersection. But since $F$ is finite, $\mathfrak{I}$ is actually closed under *arbitrary* intersections (as well as arbitrary unions). Consequently any $x \in F$ has a smallest neighborhood, namely the intersection,

$$\Lambda(x) = \cap\{A \in \mathfrak{I} \mid x \in A\} \tag{3.1}$$

of all the open sets containing it. This association of subsets of $F$ to elements of $F$ allows one to convert the natural ordering on subsets to a relation on elements. Denoting this relation by an arrow, we have the definition,

$$x \to y \Leftrightarrow \Lambda(x) \subset \Lambda(y) \tag{3.2}$$

Since in any case $x \in \Lambda(x)$, this is equivalent to

$$x \to y \Leftrightarrow x \in \Lambda(y) \tag{3.3}$$

which can also be read as saying that every open set containing $y$ contains $x$ as well, i.e., that $y \in \overline{\{x\}}$, the closure of the set $\{x\}$. This last interpretation shows that our arrow notation is literal: $x \to y$ iff the constant sequence $x$ in fact *converges* to the point $y$ in the topology $\mathfrak{I}$.

Now it is immediate from (3.2) that $\to$ is transitive and reflexive:

$$x \to x$$

$$x \to y \to z \Rightarrow x \to z$$

Conversely, given any relation $\to$ with these properties, one acquires a topology on $F$ by setting, for $x \in F$,

$$\Lambda(x) := \{y \in F \mid y \to x\} \tag{3.4}$$

and defining a subset $A \subset F$ to be open iff it is a union of sets of the form $\Lambda(x)$, i.e., iff $x \to y \in A \Rightarrow x \in A$. It is clear that the correspondences we have just set up between topologies $\mathfrak{I}$ and relations $\to$ are inverses of each other.

So for a finite set $F$ the notion of topology is equivalent to that of a transitive, reflexive relation (sometimes called "preorder"). When will such a relation on $F$ give rise to a topology that makes $F$ a $T_0$-space? Well, $F$ *fails* to be $T_0$ when there are points $x \neq y$ for which every open set containing $x$ contains $y$ and vice versa, thus, when $\Lambda(x) = \Lambda(y)$, or equivalently when a "circular" order relation $x \to y \to x$ occurs. Reexpressing this in a positive way, we conclude that $(F, \mathfrak{I})$ is $T_0$ if and only if the relation $\to$ is a *partial order*. In that case $F$ becomes a *partially ordered set* or "*poset*".

Because only $T_0$-spaces will interest us, I will henceforth always assume that $\rightarrow$ is a partial ordering. I will also use the order-theoretic language interchangeably with the topological language, than which it is often more convenient. In particular the order-dual of (3.4),

$$V(x) := \{y \in F \mid x \rightarrow y\} \tag{3.5}$$

is just the closure $\overline{\{x\}}$. Then, extending the notation $\Lambda$, $V$ to subsets in the obvious way, we have for $A \subset F$

$$A \text{ is closed} \Leftrightarrow A = V(A)$$

$$A \text{ is open} \Leftrightarrow A = \Lambda(A)$$

[Note, however, that although $V(A)$ is the closure of $A$, $\Lambda(A)$ is in general not its interior.] Finally, a map $f: F_1 \rightarrow F_2$ is *continuous* iff it is *order-preserving*, i.e., iff $x \rightarrow y \Rightarrow f(x) \rightarrow f(y)$. (Proof: continuity just says $f$ preserves convergence of sequences.)

Until now we have always been taking $F$ to be a finite set. However, it is clear that the spacetime of a spatially infinite universe, or even of a closed universe infinite in time, can properly correspond only to an $F$ with an infinite number of elements. As long as such an $F$ was, for example, derived from a *locally* finite open cover of the spacetime, its topology would still be equivalent to the partial ordering $\rightarrow$ defined, as above, by (3.1) and (3.2). Indeed, one can see that this equivalence obtains for any topological space whose subsets $\Lambda(x)$, defined by (3.1), are all open [see Bourbaki (1966) and Stanley (1986) for this relation between order and topology]. Conversely, any poset whatsoever can be obtained in this way from some $T_0$-topological space. However, not any poset is reasonable as a finitary[5] spacetime topology. Rather, as suggested by considering the properties of open coverings, one can probably regard as "finitary" only those posets $F$ for which the sets $\Lambda(x)$ and $V(x)$ defined by (3.4) and (3.5) are all finite.[6] Certainly this condition will be fulfilled by any poset derived from the covering of a manifold by a locally finite collection of bounded open sets.[7]

[5] I am avoiding the word "discrete" because it already has a technical meaning in the topological context.

[6] The term "locally finite" would also be appropriate; but it already has been defined, both for posets and for topological spaces. Unfortunately, those two definitions disagree with each other, and also with what I am here calling finitary. In particular, a poset is called locally finite iff $V(x) \cap \Lambda(y)$ is finite for all $x, y$, a usage which is appropriate when the partial ordering has a causal, rather than a directly topological interpretation (Bombelli *et al.*, 1987; Sorkin, 1990).

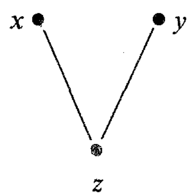
[7] A covering of $S$ is *locally finite* iff every $x \in S$ has a neighborhood that meets only a finite number of the covering sets. By *bounded* I mean a set whose closure is compact.

One of the features that make finite (or locally finite) topological spaces attractive as finitary analogs of spacetime is that, by definition one can immediately define homology and homotopy groups for them. Usually, when one passes from the continuum to some discrete analog, one loses all topological information. In lattice gauge theories, for example, it is difficult—if not impossible—to identify instantons or monopoles, because the winding numbers in terms of which these objects are defined become meaningless on a lattice. In the present case, however, the continuum definitions do carry over to the finitary case, and one can ask how well the resulting invariants agree with their continuum values. The answer (Aleksandrov, 1956) is that the agreement is virtually perfect! To conclude this section, let us look at a few examples of finite topological spaces derived from coverings, seeing in particular how a nontrivial first homotopy group arises for one of them.

Our first example is the space $F(\mathcal{U})$ of Fig. 1. This space has for elements the three equivalence classes $x = A \backslash B$, $y = B \backslash A$, and $z = A \cap B$ related as follows:

$$x \leftarrow z \rightarrow y$$

If represented in the manner customary for posets, it looks like




where rising lines represent arrows. We need not consider any homology or homotopy groups in this case, since they all vanish.

The next example, also homotopically trivial, merely generalizes the previous example from one to two dimensions. In Figure 3a, three open disks cover a portion $S$ of the plane, and $\mathcal{U}$ will be the topology for $S$

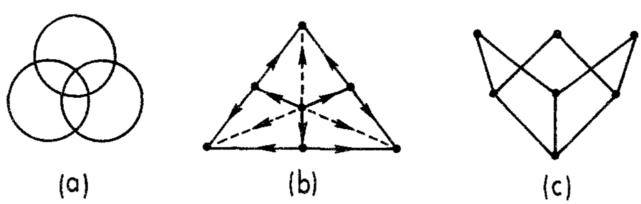



**Fig. 3.** (a) A covering of a portion of the plane. (b) The associated poset. (c) The "Hasse diagram" of the poset.

generated by the disks. Clearly, $F(\mathcal{U})$ will have seven elements, each being one of the regions into which the disks divide $S$. In Figure 3b the associated poset is indicated by arrows (the dashed arrows being redundant in the sense that they are implied by the others); and in Figure 3c the same poset is given as a "Hasse diagram", the solid lines corresponding to the solid arrows (the "links") of Figure 3b.

Our third example derives from the circle $S^1 = (\mathbf{R} \bmod 1)$ with $\mathcal{U}$ being the subtopology generated by the open covering, $\{A = (-2/6, 2/6),\ B = (1/6, 5/6),\ C = (1/6, 2/6)\}$. The associated finite space $F = F(\mathcal{U})$ has four elements, namely $x := A \backslash B = [-1/6, 1/6]$, $y := B \backslash A = [2/6, 4/6]$, $u := C$, and $v := (A \cap B) \backslash C = (4/6, 5/6)$ bearing the order relations of Figure 4.

What is the first homotopy group of this space? Well, by definition (and the definition is meaningful for arbitrary topological spaces, not excluding finite ones!) $\pi_1(F)$ is the set of homotopy classes of continuous maps $f: [0, 1] \to F$ such that $f(0) = f(1) = x$ (say).

At first glance, one might wonder how any nonconstant map of $[0, 1]$ into $F$ can be continuous, and might consequently be tempted to conclude that $\pi_n$, $H_n$, etc., are all trivial. However, continuity requires only that $f^{-1}(X)$ be open for every open $X \subset F$. Consider, then, the mapping $f$ given for $\lambda \in [0, 1]$, by

$$f(\lambda) = \begin{cases} x & \text{if } \lambda = 0 \text{ or } \lambda = 1 \\ u & \text{if } 0 < \lambda < 1/2 \\ y & \text{if } \lambda = 1/2 \\ v & \text{if } 1/2 < \lambda < 1 \end{cases} \tag{3.6}$$

I claim that $f$, which intuitively "winds once around the square" in Figure 4, is continuous. To verify this, let us consider, for example, the open sets $\{u\}$ and $\{y, u, v\}$, whose preimages by $f$ are, respectively, $f^{-1}(u) = (0, 1/2)$ and $f^{-1}(\{y, u, v\}) = (0, 1)$, both of which are manifestly open in $[0, 1]$. In

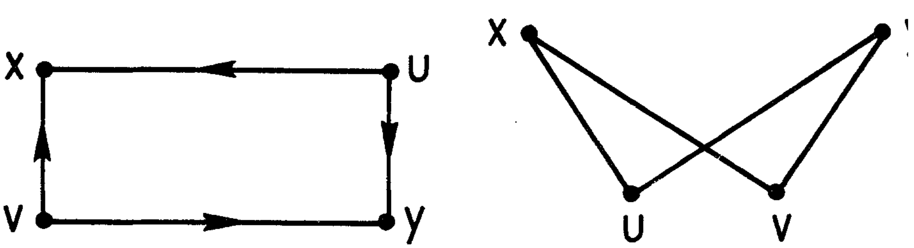



**Fig. 4.** A poset derived from the circle and its Hasse diagram.

the same way one can check that $f^{-1}(X)$ is open for the other two (nontrivial) open subsets of $F$, $\{x, u, v\}$ and $\{v\}$. The map $f$ is therefore continuous, and may be called a *curve* in $F$. [Along these lines one can show that, more generally, a curve in any finite topological space $F$ is in effect merely a sequence $(x_k)$ of elements of $F$, such that, for all $k$, $x_k$ and $x_{k+1}$ are *related* by the order $\to$ (i.e., $x_k \to x_{k+1}$ or vice versa).] With somewhat more effort, one can see by analyzing continuity of functions from $[0,1]\times[0,1]$ to $F$ that the map $f$ defined by (3.6) is not contractible, and in fact generates $\pi_1(F)$, which consequently is $\mathbf{Z}$. Thus, $\pi_1(F)=\pi_1(S^1)$, an instance of the "virtually perfect agreement" mentioned above.

Our fourth example generalizes the one just discussed from $S^1$ to an arbitrary $n$-sphere $S^n$. It is a poset containing $2(n+1)$ elements $\{x_k^{(a)} \mid a=1,2;\ k=0,\ldots,n\}$ related according to: $x_j^{(a)} \to x_k^{(b)}$, $a, b=1,2; j<k$; as illustrated in Figure 5. Diagrammatically, it consists of $n$ copies of the poset of Figure 4, stacked on top of one another, for a total of $(n+1)$ levels. If the reader has not seen these spaces before, he or she may enjoy constructing the finite open coverings of $S^n$ from which they can be derived.

Our final example is a poset with 13 elements and 24 links which may be obtained from a covering of $RP^2 = S^2/\mathbf{Z}_2 =$ the projective plane. It is depicted in Figure 6; wherein, for clarity, only the links have been shown.

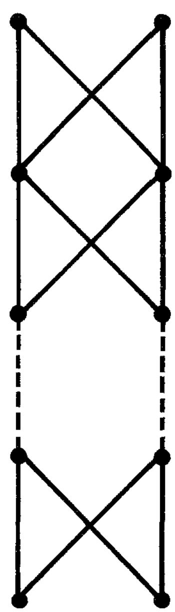
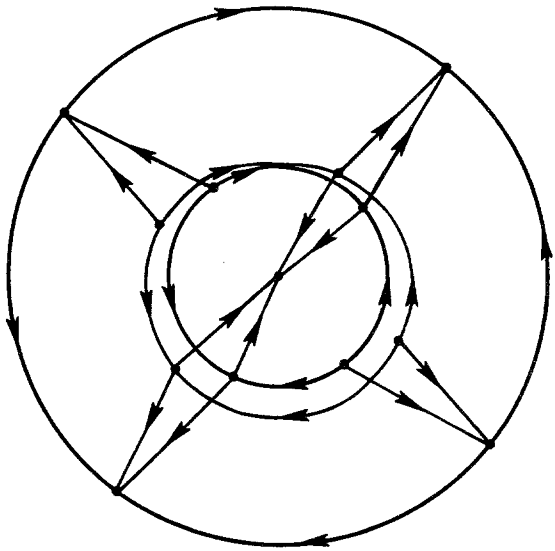
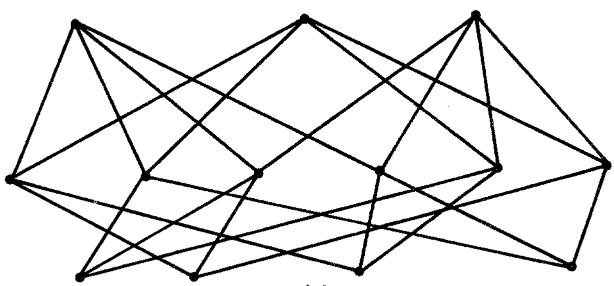


**Fig. 5.** A poset derived from the $n$-sphere.



(a)



(b)

**Fig. 6.** (a) The representation, in terms of its links, of a poset derived from the projective plane $RP^2$. (b) The same poset rendered as a Hasse diagram.

The other 12 relations may (like the dashed relations in Figure 3b) be deduced from the links by transitivity.

In this case the curve defined by the outer circle in Figure 6a generates $\pi_1(F) = \pi_1(RP^2) = \mathbf{Z}_2$. In a way that can be made precise, the fact that $F$ is only doubly connected shows up in the figure as the fact that the twice-traversed outer circle can be contracted onto the intermediate loop of eight elements, and from there onto the central point.

I hope the discussion of this section has illustrated the characteristic way that topological issues get reformulated when one deals with finite spaces. Just as analytical geometry introduces an algebraic-arithmetic language into topology and the simplicial complex introduces a combinatorial language, one may say that the finite topological space introduces an order-theoretic way of speaking about manifolds and other spaces of interest to physics. Although such an approach to topology seems barely to have been explored so far, there are still several items which would have to be included in a more complete introduction to finite topological spaces than I am presenting here. Chief among them, probably, is the existence of a construction running in the inverse direction to the process (continuous space $S$) $\to$ (finitary space $F$) described in Section 2. The construction in question [explained in detail in McCord (1966), Stong (1966), and Stanley (1986)] produces from $F$ a simplicial complex $\Sigma(F)$, sometimes called the "order complex" of $F$. In each of the examples discussed above $\Sigma(F(\mathcal{U}))$ reproduces the manifold we would expect [for example, $\Sigma(F)=RP^2$ in the last example], but unfortunately this is not true in general, even for sufficiently fine covers $\mathcal{U}$. [A similar problem exists with the "nerve" construction (Aleksandrov, 1956), which also fails to produce a simplicial complex homeomorphic to the original manifold.]

Finally, I cannot refrain from mentioning two small applications of the order-theoretic way of thinking to topology. The first concerns triangulation of product spaces, the second is a method to compute Euler characteristics. For the first, let $S_1=\Sigma(F_1)$ and $S_2=\Sigma(F_2)$ be two simplicial complexes derived from posets. It is well known that there is no canonical way to express the Cartesian product of two simplicial complexes as a third such complex. On the other hand, the product $F_1\times F_2$ of two posets *is* naturally again a poset [by defining $(x_1,x_2)\to(y_1,y_2)\Leftrightarrow x_1\to y_1$ and $x_2\to y_2$]. Then $\Sigma(F_1\times F_2)$ furnishes a natural triangulation of $S_1\times S_2$. The method for computing the Euler characteristic $\chi(F)$ $[=\chi(\Sigma(F))]$ is slightly too long to explain here (Stanley, 1986), but it follows easily from the two identities $\chi(\Lambda(x))=0$ for any $x\in F$ and $\chi(A\cup B)=\chi(A)+\chi(B)-\chi(A\cap B)$ for any subsets $A, B\subset F$.

## 4. CONVERGENCE OF $F(\mathcal{U})$ TO $S$

### 4.1. The Inverse System $\mathcal{K}$

In Section 2 we found that a finite subtopology $\mathcal{U}$ on the topological space $S$ (or what is essentially the same thing, a finite open cover of $S$) gives rise to a finite space $F(\mathcal{U})$ which "approximates" $S$ to a certain accuracy. In Section 3 we found that $F(\mathcal{U})$ can be described equivalently in terms of the order relation we denoted by "$\to$"; and I mentioned that the equivalence continues to hold for a wider class of open covers, which

I called "finitary". In the present section we will find that as $\mathcal{U}$ acquires more (and finer) open sets, the approximation of $S$ by $F(\mathcal{U})$ becomes exact. Namely, we will find that the $F(\mathcal{U})$ form an inverse system, and this system *converges to S* in a certain sense. Without some result of this sort, it would be difficult to entertain either the idea that posets are a more fundamental replacement for the continuum, or the idea that they can be useful in a technical way as a kind of "lattice spacetime" used to study, for example, the renormalization group properties of "Regge calculus".

The type of space $S$ that we would like to approximate is probably a manifold, or something very close to a manifold. However, the approximation theorem we will actually prove is not limited to such a special case. Let us begin, then, with an arbitrary topological space $S$ and a collection $\{\mathcal{U}\}$ of open covers of $S$. We will assume that $\{\mathcal{U}\}$ is *directed* in the sense that for any two of its elements $\mathcal{U}_1, \mathcal{U}_2$ there is a third $\mathcal{U}_3$ such that $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathfrak{I}(\mathcal{U}_3)$, where[8] $\mathfrak{I}(\mathcal{U})$ is the topology generated by $\mathcal{U}$ (the unions of finite intersections of elements of $\mathcal{U}$).

We have already seen how to associate to each $\mathcal{U} \in \{\mathcal{U}\}$ a $T_0$ topological space $F(\mathcal{U})$ and a continuous surjection $f(\mathcal{U})$ of $S$ onto $F(\mathcal{U})$. For notational convenience we can label the elements of $\{\mathcal{U}\}$ with an index $j$, writing then $f_j$ and $F_j$, respectively, for $f(\mathcal{U}_j)$ and $F(\mathcal{U}_j)$. We have then

$$f_j: \quad S \to F_j$$

Now let us define a partial ordering on our indices expressing the relation of inclusion among the $\mathfrak{I}(\mathcal{U})$:

$$j \leq k \Leftrightarrow \mathcal{U}_j \subset \mathfrak{I}(\mathcal{U}_k)$$

Thus, $\leq$ expresses the notion of refinement we are using; and the condition that $\{\mathcal{U}\}$ is directed translates to the statement that for every pair of indices $i, j$ there exists an index $k$ greater than both of them.[9]

Now we would like to prove that "$F_k \to S$ as $k \to \infty$", but to interpret such a statement literally would presuppose the existence of a topology on the "space of all topological spaces", with respect to which the convergence $F_k \to S$ could be defined. Unfortunately no such topology exists (to my knowledge). Instead we will use a different notion of limit, which pertains not just to a sequence (more exactly a "net") of spaces $F_k$, but to an "inverse system" of spaces $F_k$ *together with* maps $f_{jk}: F_k \to F_j$, defined whenever $j \leq k$. In order to obtain these maps, consider Figure 7, in which $j \leq k$. As explained in Section 2, the universal mapping problem indicated by Figure 7 will

[8]In previous sections we have assumed that $\mathcal{U}$ already is a subtopology: $\mathcal{U} = \mathfrak{I}(\mathcal{U})$. Here we relax this harmless assumption for greater ease of application.

[9]A *sequence* $\mathcal{U}_1 \leq \mathcal{U}_2 \leq \cdots$ is, of course, the most important case of a directed set. However, it is not sufficiently general for certain cases, such as the system of *all* finite open covers of $S$.

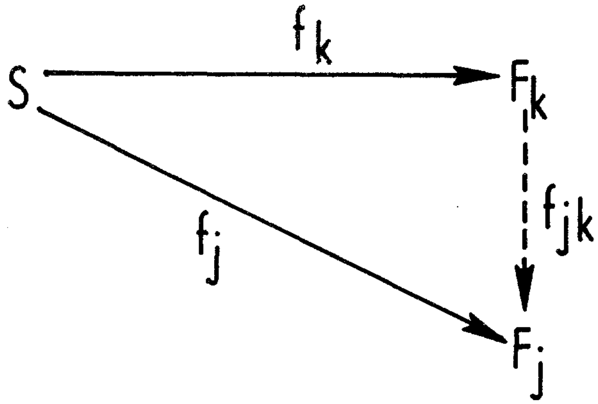



**Fig. 7.** Defining the map $f_{jk}$.

furnish a unique $f_{jk}$ provided only that $F_j$ is $T_0$ and that $f_j$ is continuous with respect to $\mathfrak{I}(\mathcal{U}_k)$.

Now in fact $F_j$ is $T_0$ by construction, while the desired continuity follows from the inclusions $f_j^{-1}(\mathfrak{I}(F_j)) \subseteq \mathfrak{I}(\mathcal{U}_j) \subseteq \mathfrak{I}(\mathcal{U}_k)$. [Here $\mathfrak{I}(F_j)$ is the topology of $F_j$, the first inclusion is true by construction, and the second merely says that $j \leq k$. In words: let $V$ be open in $F_j$; then $f_j^{-1}(V)$ is open with respect to the $\mathcal{U}_j$-topology by the definition of $F_j$, whence it is also open with respect to the finer $\mathcal{U}_k$-topology.] Because solutions of universal mapping problems are by definition unique, it follows immediately that

$$f_{ij} f_{jk} = f_{ik} \tag{4.1}$$

for $i \leq j \leq k$. Thus, all the requirements for an inverse system of spaces and maps are fulfilled.

*Definition.* $\mathcal{K}$ is the inverse system of $T_0$-spaces $F_k$ and continuous maps $f_{jk}$.

Now any inverse system of topological spaces $F_j$ and maps $f_{jk}$ has a so-called "inverse limit", which is a certain topological space $F_\infty$ together with maps $f_{j\infty}: F_\infty \to F_j$, that can be regarded as the limits of the $f_{jk}$ as $k \to \infty$. After recalling the definition of this limit, we will prove a series of lemmas, which taken together will establish that in all cases of interest $F_\infty$ is essentially the space $S$ with which we began. I say "essentially" because—unfortunately—$F_\infty$ will not be precisely $S$, but a non-Hausdorff space containing the latter as a dense subspace. This discrepancy can be overcome, as we will see, either by a slight modification of the notion of inverse limit, or else just by realizing that $S$ itself can be recovered from the unmodified $F_\infty$ as the set of *closed* points thereof.

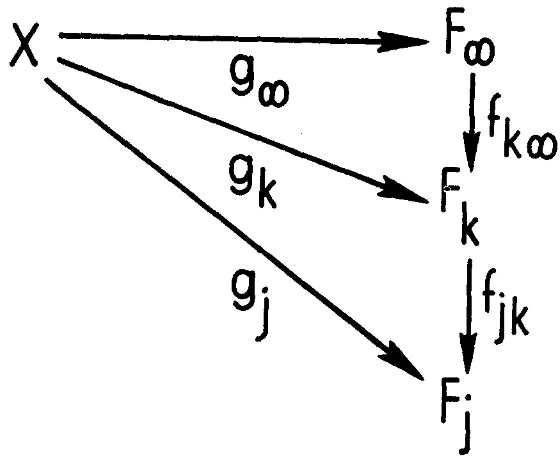



**Fig. 8.** Defining the inverse limit.

The notion of inverse limit (Bourbaki, 1968) can be defined most compactly as the solution of yet another universal mapping problem set up as follows (see Figure 8). Let $X$ be an arbitrary topological space and let $\{g_k : X \to F_k\}$ be a system of maps[10] coherent in the sense that $f_{jk} g_k = g_j$ whenever $j \leq k$. Then there is a unique map (see footnote 10) $g_\infty : X \to F_\infty$ such that $f_{k\infty} g_\infty = g_k$, for all $k$. It turns out that $F_\infty$ and the maps $f_{k\infty}$ can be constructed explicitly: An element $x \in F_\infty$ is an arbitrary coherent system of elements $x_k \in F_k$ [where coherence means, of course, that $x_j = f_{jk}(x_k)$ whenever $f_{jk}$ is defined (i.e., whenever $j \leq k$)] and then $f_{k\infty}(x)$ is taken to be merely $x_k$. With these definitions (4.1) is guaranteed by the coherence of the system $x$. Finally, $F_\infty$ is endowed with the weakest topology compatible with the continuity of all the $f_{k\infty}$; a basis for it is given by the sets $f_{k\infty}^{-1}(V)$, $V$ open in $F_k$.

From this construction it is easy to see that $F_\infty$ will be $T_0$ if all of the spaces $F_k$ are $T_0$. Let us also note, for future reference, that all the $f_{jk}$ are surjective, as follows directly from the surjectivity of $f_j$ and the equality $f_j = f_{jk} f_k$.

### 4.2. Properties of the Limit

Let us return now to the specific inverse system $\mathscr{K}$ of interest to us and examine the maps $f_k : S \to F_k$ in light of the universal mapping problem by which $F_\infty$ is defined. Identifying $S$ and $f_k$ with $X$ and $g_k$ of the universal mapping problem, we acquire a unique map $f_\infty : S \to F_\infty$, for which

[10]Recall that in this paper any map between topological spaces is by convention continuous.

$f_{k\infty}(f_\infty(s)) = f_k(s)$. Let us see how far we can go toward proving that $f_\infty$ is a homeomorphism (which would mean that $F_\infty = S$). So far we have placed no restrictions at all on the space $S$ or the coverings $\mathcal{U}$ (other than that $\{\mathcal{U}\}$ is a directed system). We will now add conditions as needed, none of which will be essentially restrictive when $S$ is, for example, a (Hausdorff) manifold.

*Lemma 4.1.* $f_\infty(S)$ is dense in $F_\infty$.

*Proof.* Let $W \subset F_\infty$ be any nonempty open set. We must find $x \in W$ and $s \in S$ such that $f_\infty(s) = x$. Now, by definition of the topology of $F_\infty$, $W$ is a union of sets of the form $f_{k\infty}^{-1}(V_k)$. Picking one of these sets, choose $x_k \in V_k$ and $s \in f_k^{-1}(x_k)$ [such an $s$ exists because $f_k$ is surjective]; and let $x = f_\infty(s)$. Then $f_{k\infty}(x) = f_{k\infty} f_\infty(s) = x_k$, whence $x \in f_\infty^{-1}(x_k) \subset f_{k\infty}^{-1}(V_k) \subset W$, as required. ■

*Lemma 4.2.* $f_\infty$ is injective if $S$ is $T_0$ and if the following condition is satisfied: For every $s \in S$ and every neighborhood $N$ of $s$, there exists an index $k$ and an element $U$ of $\mathcal{U}_k$ such that $s \in U \subset N$.

(Notice that this last condition is necessarily satisfied if $\mathcal{U}_k$ contains "enough small open sets" in any plausible sense.)

*Proof.* Let $s_1$ and $s_2$ be distinct elements of $S$. Since the latter is $T_0$, there is an open set $W$ containing (say) $s_1$ but not $s_2$. By hypothesis there exist $k$ and $U \in \mathcal{U}_k$ for which $s_1 \in U \subset W$, whence $\mathcal{U}$ distinguishes $s_1$ from $s_2$. Hence, by the definition of $F_k$ as a $T_0$-quotient, $f_k(s_1) \neq f_k(s_2)$, whence $f_{k\infty} f_\infty(s_1) \neq f_{k\infty} f_\infty(s_2)$, whence $f_\infty(s_1) \neq f_\infty(s_2)$. ■

Taken together the first two lemmas tell us that $S$—or more precisely $f_\infty[S]$—is embedded in $F_\infty$ as a dense subset; but what about the other points of $F_\infty$? We will see that for every such "extra" $y \in F_\infty$, there exists some $x \in f_\infty[S]$ to which $y$ is "infinitely close". Thus, the extra points are in this sense "superfluous", and may be eliminated in one of the ways mentioned above.

By saying that $y$ was "infinitely close" to $x$, I meant that $y$ belongs to every neighborhood of $x$. For brevity let us write this relationship as $y \in N(x)$, by defining, for an arbitrary topological space $Z$ and element $z \in Z$, the set $N(z)$ as the intersection of all the open sets containing $z$:

$$N(z) := \cap\{U \mid z \in \mathring{U} \subset Z\}$$

[Notice that $N(z) = \Lambda(z)$ if $Z$ is finitary.] What we want to prove can thus be stated as follows:

*Lemma 4.3.* The sets $N(x)$ for $x \in f_\infty[S]$ cover $F_\infty$ if, for all $k$, the elements of $\mathcal{U}_k$ are bounded (have compact closure). In particular, the conclusion holds if $S$ itself is compact.

To simplify the proof of this lemma, let us introduce one more definition and an auxiliary result.

*Definition.* For $x \in F_\infty$, let $x(k) = f_k^{-1} f_{k\infty}(x)$.

In other words, $x(k)$ is the equivalence class in $S$ to which $f_{k\infty}(x)$ corresponds. If, as in the concrete construction of $F_k = F(\mathcal{U}_k)$ given in Section 2, we actually identify $f_{k\infty}(x)$ with this equivalence class, then we can also write $x(k) = f_{k\infty}(x)$.

*Lemma* 4.4. $k \geq j$ and $x \in F_\infty \Rightarrow x(k) \subset x(j)$.

The lemma, and therefore its proof, is essentially a matter of notation:

*Proof.* $f_{k\infty}(x) \subset f_{jk}^{-1} f_{jk}(f_{k\infty}(x)) = f_{jk}^{-1} f_{j\infty}(x)$. Therefore,

$$x(k) = f_k^{-1} f_{k\infty}(x) \subset f_k^{-1} f_{jk}^{-1} f_{j\infty}(x) = (f_{jk} f_k)^{-1} f_{j\infty}(x) = f_j^{-1} f_{j\infty}(x) = x(j). \quad \blacksquare$$

*Proof of Lemma 4.3.* Fix an element $y$ of $F_\infty$. We must demonstrate that $y$ belongs to $N(f_\infty(s))$ for some $s \in S$. Consider the family $\mathcal{F}$ whose members are the sets $y(j)$. I claim $\mathcal{F}$ has the "finite intersection property", meaning that every finite subfamily of $\mathcal{F}$ has a nonempty intersection. To prove this, let $B := y(a) \cap y(b) \cap \cdots \cap y(c)$. Since $\{\mathcal{U}\}$ is directed, there exists some index $k$ greater than all the indices $a, b, \ldots, c : k \geq a, k \geq b, \ldots, k \geq c$. Then, by Lemma 4.4, $y(k) \subset B$, which means that $B$ cannot be empty (the case where $S$ itself is empty can obviously be ignored).

It follows *a fortiori* that the compact sets $\overline{y(k)}$ *also* enjoy the finite intersection property, and hence (e.g., Kelley, 1955) that there exists a point $s \in S$ common to *all* the sets $\overline{y(k)}$. We will take $x = f_\infty(s)$ and prove that $y \in N(x)$.

To prove this, it suffices to show that $y$ is contained in any neighborhood of $x$ of the form $f_{k\infty}^{-1}(W)$, $W$ open in $F_k$; so let $f_{k\infty}^{-1}(W)$ be such a neighborhood of $x$. Now on one hand $f_{k\infty}(x) \in W$, obviously. On the other hand, we know $s \in \overline{y(k)}$ by construction, whence

$$f_{k\infty}(x) = f_{k\infty} f_\infty(s) = f_k(s) \in f_k \overline{(y(k))} \subset \overline{f_k(y(k))} = \overline{f_k f_k^{-1} f_{k\infty}(y)} = \overline{f_{k\infty}(y)}$$

or $f_{k\infty}(x) \in \overline{f_{k\infty}(y)}$. Therefore (by the definition of closure) $W$, being a neighborhood of $f_{k\infty}(x)$, must also contain $f_{k\infty}(y)$; i.e., $f_{k\infty}(y) \in W$, or $y \in f_{k\infty}^{-1}(W)$, as required. $\blacksquare$

If we interpret the relation $y \in N(x)$ as meaning that $y$ is "infinitely close to $x$", then we have shown that every point in $F_\infty$ either can be identified with an element of $S$, or is infinitely close to a point which can.

However, we lack, as yet, a criterion to distinguish the first case from the second, and thereby to fully recover $S$ directly from the inverse system $\mathcal{K}$ of approximating spaces $F_k$ and maps $f_{jk}$. The following lemma provides such a criterion, showing that $\mathcal{K}$ contains complete information from which $S$ and its topology can be entirely reconstructed.

*Lemma 4.5.* If $S$ is a $T_1$ space and the conditions of Lemmas 4.2 and 4.3 are satisfied, then $f_\infty$ imbeds $S$ in $F_\infty$ as the set of closed points of $F_\infty$.

*Proof.* We wish to prove that $x \in f_\infty[S]$ if and only if $x$ is closed: $\overline{\{x\}} = \{x\}$. First, suppose $x$ is closed. We know from Lemma 4.3 that $x \in N(y)$ for some $y \in f_\infty[S]$. But $x \in N(y)$ is equivalent to $y \in \overline{\{x\}} = \{x\}$, which means $x = y$. Thus, $x \in f_\infty[S]$ since $y \in f_\infty[S]$.

Conversely, let $x = f_\infty(s)$, and let $y \neq x$ be any other element of $F_\infty$. We must show $y \notin \overline{\{x\}}$. Now either $y \in N(x)$ or not. If it is, then $x \notin N(y)$, since otherwise $F_\infty$ would not be $T_0$, contrary to what we saw in Section 4.1. But $x \notin N(y) \Rightarrow y \notin \overline{\{x\}}$.

If $y \notin N(x)$, then, by Lemma 4.3 there exist $z \neq x$ and $t \in S$ such that $z = f_\infty(t)$ and $y \in N(z)$. Now $z \neq x \Rightarrow t \neq s$, whence, since $S$ is $T_1$, there exists[11] an open set $U$ in $S$ containing $t$, but not $s$. By replacing $U$ with a smaller set if need be, we can suppose (by the condition of Lemma 4.2) that $U \in \mathcal{U}_k$ for some index $k$. Then let $W = f_{k\infty}^{-1} f_k[U]$, which is open because $f_k[U]$ is open and $f_{k\infty}$ continuous. On one hand, $t \in U$ implies that $z = f_\infty(t) \in f_{k\infty}^{-1} f_{k\infty} f_\infty(t) = f_\infty^{-1} f_k(t) \subset f_{k\infty}^{-1} f_k[U] = W$, whence $z \in W$. On the other hand, we can prove that $x \notin W$:

$$x \in W = f_{k\infty}^{-1} f_k[U] \Leftrightarrow f_{k\infty}(x) \in f_k[U] \Leftrightarrow f_{k\infty} f_\infty(s) \in f_k[U]$$

$$\Leftrightarrow f_k(s) \in f_k[U] \Leftrightarrow s \in U$$

(by the definition of $f_k$), whence $x \notin W$ because $s \notin U$. Then $z \in W \Rightarrow y \in N(z) \subset W$, which together with $x \notin W$, implies $y \notin \overline{\{x\}}$. ■

*Remark.* The condition that $S$ be $T_1$ is certainly obeyed if $S$ is a (not necessarily Hausdorff) manifold. From the lemmas follows immediately:

*Theorem 1.* Let $S$ be $T_1$, let $\{\mathcal{U}\}$ fulfill the "fineness" condition of Lemma 4.2, and let each cover $\mathcal{U}_k$ consist entirely of sets $U$ with compact closure. Then $f_\infty: S \to F_\infty$ densely embeds $S$ in $F_\infty$ as the subspace of closed points.

[11] A space is $T_1$ iff its points are closed, equivalently iff for any two distinct points, the first has a neighborhood excluding the second.

Given this result, we could perhaps tinker with the definition of inverse limit so that it would directly produce $S$ rather than yielding extraneous points which then have to be excluded by fiat. In this connection it may be relevant—and it is in any case interesting—that there exists another characterization of $\hat{F}_\infty := f_\infty[S]$, one expressed purely in order-theoretic language. In fact, the coherent systems $\{x_k\}$ from which one may construct $F_\infty$ also define an inverse limit in the category of posets, if ordered according to

$$x < y \Leftrightarrow \forall k, \quad x_k < y_k$$

where the order relation on the $F_k$ has been renamed from "$\rightarrow$" to "$<$". With this definition we can alternately characterize $\hat{F}_\infty$ as *the set of maximal elements of* $F_\infty$, with respect to the order $<$.

Let me also state without proof a strengthened version of Lemma 4.3 that applies when $S$ is Hausdorff.

*Lemma 4.6.* If $S$ is Hausdorff and fulfills the conditions of Lemmas 4.2 and 4.3, then the sets $N(f_\infty(s))$ *partition* $F_\infty$.

Using this, one can prove that $F_\infty$ *is homotopic to* $S$ [in fact $S$ is a (strong) deformation retract of $F_\infty$]. This is not needed for reconstructing $S$ (as we have seen), but it might be useful in relating the algebraic-topology invariants of $S$ (its homology groups, etc.) to those of the $F_k$.

Finally, we *will* prove a result which would seem to be needed in order that one can speak in a *physical* sense of the convergence of $\mathscr{K}$ to $S$. This result says roughly that the elements of the $F_k$ correspond to sets in $S$ that *become small* as $k \rightarrow \infty$. It is all the more interesting because the analogous assertion does *not* hold for inverse systems of simplicial complexes and simplicial maps.

A theorem like the one we will prove could be formulated for any inverse system of topological spaces, but for simplicity we will continue to limit ourselves to systems $\mathscr{K}$ that arise from systems of coverings of a given space $S$. In fact, we will eliminate as many inessential complications as possible by assuming that $S$ is compact-Hausdorff, and will also assume from the outset that $\{\mathscr{U}\}$ obeys the condition of Lemma 4.2 and contains only finite covers $\mathscr{U}$.

Now let us think of $S$ as some physical continuum, and let us regard the elements of a given $F_j$ as approximations to the points of $S$, or, more appropriately (since it can be obtained from $\mathscr{K}$ without direct reference to the system of coverings $\mathscr{U}$), of $\hat{F}_\infty := f_\infty[S]$. Let us define further $\hat{f}_{j\infty} := f_{j\infty}|\hat{F}_\infty$ and consider the sets $\hat{f}_{j\infty}^{-1}(x)$, $x \in F_j$. If a given $F_j$ is to furnish a "good" approximation to $\hat{F}_\infty$, then the sets $\hat{f}_{j\infty}^{-1}(x)$ should be approximately point-like, i.e., they should be "small" subsets of $\hat{F}_\infty$. Of course, we cannot determine smallness in an absolute sense except with respect to some metric, but we can meaningfully require that the tiling of $\hat{F}_\infty$ given by the $\hat{f}_{j\infty}^{-1}(x)$ *become arbitrarily fine* as $j \to \infty$. Let us agree to judge this in terms of the following criterion.

*Criterion.* Let $\Phi_j$ be an indexed family of subsets of a compact space $Z$, with indices directed by the partial ordering $\leq$. We will say that $\Phi_j$ *becomes arbitrarily fine* in $Z$ iff for each covering $\mathscr{V}$ of $Z$ by a finite number of open sets $V_\alpha$, there exists $k$ such that each family $\Phi_j$ with $j \geq k$ is "$\mathscr{V}$-fine" in the sense that each $\phi \in \Phi_j$ is contained in some $V_\alpha \in \mathscr{V}$.

After this lengthy preamble, and under the assumptions set out above, we can state the following theorem.

*Theorem 2.* The family $\Phi_j \equiv \{\hat{f}_{j\infty}^{-1}(x) \mid x \in F_j\}$ becomes arbitrarily fine in $\hat{F}_\infty$.

*Proof.* Let $\{V_\alpha \mid \alpha = 1, \ldots, n\}$ be a fixed open cover of $\hat{F}_\infty$, which we will identify with $S$, in order to avoid proliferation of notation. By the condition of Lemma 4.2 we can find for each $\alpha$ and each $x \in V_\alpha$ an index $j$ and a $U \in \mathscr{U}_j$ such that $x \in U \subset V_\alpha$. The totality of sets $U$ found in this way cover $S$ (obviously), whence (since $S$ is compact) some finite selection $\mathscr{C} = \{U_1, \ldots, U_N\}$ of them suffices to cover $S$. Now let $U_i \in \mathscr{U}_{j_i}$, and find a $k$ which is simultaneously greater than $j_1, \ldots, j_N$ (which is possible because $j$ is a directed index). Then by the definition of $\leq$, $U_i \in \mathfrak{I}(\mathscr{U}_k)$ for $i = 1, \ldots, N$, or in other words, $\mathscr{C} \subset \mathfrak{I}(\mathscr{U}_k)$, and *a fortiori* $\mathscr{C} \subset \mathfrak{I}(\mathscr{U}_j)$ for any $j \geq k$. But if $\mathscr{C} \subset \mathfrak{I}(\mathscr{U}_j)$, then the equivalence classes of $S$ defined by $\mathscr{U}_j$ are smaller than those defined by $\mathscr{C}$; that is, each set $f_j^{-1}(x)$ for $x \in F_j$ is included in some $U_i \in \mathscr{C}$. Since in addition each $U_i \in \mathscr{C}$ is contained in one of the sets $V_\alpha$, we conclude that: $\forall j \geq k$, $\forall x \in F_j$, $\exists \alpha$, $f_j^{-1}(x) \subset V_\alpha$. This completes the proof, since our identification of $S$ with $\hat{F}_\infty$ means that $f_j$ and $f_{j\infty}$ have also been identified. ■

## 5. FURTHER COMMENTS

In the preceding sections, we have encountered several suggestions that the finite topological space (or poset) possesses a structure suitable for approximating bounded portions of physically important manifolds. In particular, we have studied the way in which an inverse system $\mathscr{K}$ of such spaces can converge to a compact space $S$. We may think of such a pair $S, \mathscr{K}$ as having arisen in more than one way. On one hand we might have begun with $S$, and then derived $\mathscr{K}$ by introducing into $S$ a system of covers of ever increasing fineness. On the other hand, $S$ might have been constructed directly as the (or as a subspace of the) limit of a preexisting system $\mathscr{K}$ of finite spaces and maps. The former way would be more appropriate if we were faced with a known space $S$ which we wanted to "finitize" for computational or other purposes. The latter would be more appropriate if we needed to actually "create" $S$ by means of $\mathscr{K}$, either because some of the spaces $F_k$ really exist on a small scale, or because we are trying to capture mathematically a concept like that of "a manifold with topological features of arbitrarily small scale". In either case we have seen that the mathematical situation is largely satisfactory: the system $\mathscr{K}$ converges to $S$, and the convergence is such that the elements of successive spaces $F_j$ correspond to smaller and smaller subsets of $S$. There are also systems of "finitary" topological spaces converging to locally-compact spaces, or to more general spaces obeying suitable regularity conditions. The only drawback of the whole scheme is that the most naturally definable limit $F_\infty$ contains, in addition to $S$, points which in general need to be eliminated by means of one or another less than elegant—but nonetheless well-defined—condition. We have seen also that the study of finite topological spaces brings about a confluence of topological, combinatoric, and order-theoretic ideas that one could only expect to be fruitful.

There is, however, another combinatorial structure that is often used to approximate physical continua such as spacetime, namely the simplicial complex. It is again true that any compact space can be realized as an inverse limit of simplicial complexes and simplicial maps (Eilenberg and Steenrod, 1952), but the realization in question has two shortcomings. First, the successive complexes do not "become fine" with respect to the limit space $S$. Second, there seems to be no way to derive a system of simplicial complexes from a system of open coverings of $S$. For although there is a complex associated to any given cover (namely its nerve), there is no unique simplicial map associated to a pair of covers one of which refines the other: we get analogs of the $F_j$, but not of the $f_{jk}$. It is not clear how severe either shortcoming is (for example, the first might be remedied by allowing a more general class of maps than just simplicial ones), but they do suggest that the finite topological space/poset offers a more thoroughly combinatorial approximation tool than the simplicial complex, as well (paradoxically?) as one with a more "operational" flavor.

In any case, there is actually a very close relationship between simplicial complexes and posets, as was alluded to in Section 3. Indeed, the latter can be construed as a special case of the former (and vice versa to some extent), via the existence of a functor (called $\Sigma$ in Section 3) taking finite topological spaces to simplicial complexes and continuous maps to simplicial maps. [Conversely, it is easy to pass from a complex $\Delta$ to a poset, by introducing a convenient open cover $\mathscr{U}$ on $\Delta$ and forming $F(\mathscr{U})$.] For this reason posets can be useful simply as an aid in dealing with simplicial complexes.[12]

Of course the most interesting question is whether posets have any role to play in fundamental physics. My belief is that they do, but that the order they express has more of a causal significance than a (directly) topological one: that the finitary structure underlying spacetime is that of a "causal set" (Bombelli *et al.*, 1987; Sorkin, 1990). Even in this case, however, it may be useful to associate certain finitary topological spaces $F$ with a given causal set $C$. If $C$ can be approximated by a globally hyperbolic spacetime manifold $M$, it is obviously important to understand how the topology of $M$ relates to the structure of $C$. One way to do so may be to consider within $C$ subsets $F$ which correspond to "thickened" Cauchy hypersurfaces in $M$. Such an $F$ is by definition still a poset, and it seems plausible that, regarded as a topological space, $F$ carries considerable information on the topology of the hypersurface in question.

## ACKNOWLEDGMENTS

The ideas expressed in this paper arose at University College, Cardiff, and were refined at the University of Chicago, The Institute for Advanced Study, and the University of Maryland at College Park. I would like to thank people at all these places for their hospitality, but especially Bernard Schutz at Cardiff.

This work was supported in part by National Science Foundation grants PHY 8700651 and PHY-86-20266.

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[12]For example, they have been used in the context of Euclidean quantum gravity to produce exchange-symmetric triangulations of $CP^2 \times CP^2$ (Hartle and Sorkin, 1986).

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